# A Low Complexity Optimal LMMSE Channel Estimator for OFDM System

Jyoti P. Patra, Bibhuti Bhusan Pradhan, and Poonam Singh

Abstract—Linear minimum mean square error (LMMSE) is the optimal channel estimator in the mean square error (MSE) perspective, however, it requires matrix inversion with cubic complexity. In this paper, by exploiting the circulant property of the channel frequency autocorrelation matrix  $R_{HH}$ , an efficient LMMSE channel estimation method has been proposed for orthogonal frequency division multiplexing (OFDM) based on fast Fourier transformation (FFT) and circular convolution theorem. Finally, the computer simulation is carried out to compare the proposed LMMSE method with the classical LS and LMMSE methods in terms of performance measure and computational complexity. The simulation results show that the proposed LMMSE estimator achieves exactly same performance as conventional LMMSE estimator with much lower computational complexity.

*Index Terms*—OFDM, Channel estimation, Circulant matrix, LMMSE estimator.

## I. INTRODUCTION

Orthogonal frequency division multiplexing (OFDM) has attracted a lot of attention due to its high spectrum efficiency, fast and easy implementation using fast Fourier transformation (FFT). It is also resilient against inter-symbol interference (ISI) which occurs due to the frequency selective fading channel. The equalization of the OFDM system solely depends on the accuracy of the channel estimation [1]. Based on comb-type pilot, the least square (LS) and minimum mean square error (MMSE) based channel estimation have been investigated in [2]. The LS estimation has low computational complexity but it has higher mean square error (MSE) due to noise distortion. To obtain better performance of the LS based estimator, several denoising strategies have been proposed in [3-5]. An eigen-select denoising threshold [3], linear filtering least square method [4], ada-boost based channel estimation method [5] are proposed for channel estimation in OFDM system. However, these estimation methods provide trade-off between performance and computational complexity. If the power delay profile (PDP) of the channel is aprior known to

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the receiver, the linear minimum mean square error (LMMSE) channel estimator is typically implemented. However, it requires cubic complexity due to matrix inversion operation. To reduce the computational complexity of LMMSE estimation, several techniques have been proposed developed for both single and multiple antennas based OFDM systems. A low rank approximation based on singular value decomposition (SVD) is proposed in [6]. Based on [6], the authors proposed two efficient channel estimation methods for orthogonal frequency division multiplexing/offset quadrature amplitude modulation (OFDM/OQAM) system in [7]. However these SVD based estimation methods require still high computational complexity as decomposing the  $R_{HH}$  matrix using SVD method itself requires cubic complexity [8]. A low complexity linear minimum mean square error estimation- expectation maximization (LMMSE-EM) based channel estimation was proposed in [9] for OFDM system with transmit diversity by exploiting the circulant property of the channel matrix. In [10], the authors approximate the LMMSE estimator using the law of large numbers to reduce the computational complexity of the channel to  $O(N \log N)$ . In [11], the authors proposed a dual-diagonal LMMSE (DD-LMMSE) channel estimation method with  $O(N \log N)$  and also derive the closed form expression of the asymptotic MSE of the DD-LMMSE. However, it achieves lower performance as compared to classical LMMSE estimator at high signal to noise ratio (SNR). In [12], the authors proposed a low complexity LMMSE channel estimator based on K terms Neumann series expansion method to avoid the matrix inversion. Although this method achieves nearly same performance as LMMSE method but the processing delay of the channel estimation method increases as it estimates the channel in an iterative manner. In [13], the authors proposed a conjugant gradient (CG) based channel estimation to achieve similar performance as LMMSE method. This method performs the channel estimation in an iterative manner and requires computational complexity of the order  $O((N_p \log N_p)G)$  where G is the number of iterations. Typically a high value of G is required to obtain the optimal performance. Therefore, this method suffers from procesing delay and higher computational complexity because of large number of iterations.

In this paper, we propose a low complexity optimal LMMSE channel estimation method for OFDM system in the frequency selective fading channel. This method exploits the unique circulant property of the channel frequency response (CFR) autocorrelation matrix  $R_{HH}$ , fast Fourier transform (FFT)

and cicular convolution theorem to estimate the channel. The proposed method is directly derived from the LMMSE method without requiring any approximation. Therefore, it provides exactly same performance as the clasical LMMSE method. The computational complexity of the proposed method is also lower as this method requires only FFT operation.

The symbols associated with matrices and vectors are denoted in boldface and underline fonts, respectively. The notation  $(.)^H$  and  $(.)^{-1}$  denote the Hermitian and inverse operation.  $\circ$  and  $\otimes$  denote the Hadamard element wise product and circular convolution operation, respectively.  $(.)_p$  denotes the position of pilot signal.

The rest of the paper is organized as follows. In the Section II, the classical LS and LMMSE channel estimation techniques are discussed. The proposed low complexity LMMSE channel estimation is presented in Section III. The performances of the proposed and conventional channel estimation techniques are compared on the basis of bit-error-rate (BER) in Section IV. Finally, Section V concludes the paper.

#### II. SYSTEM MODEL

Let, consider an OFDM system with N number of subcarrier. After some signal manipulation such as addition of cyclic prefix (CP), removal of CP, inverse fast Fourier transformation (IFFT) and FFT operation, the received signal vector in the frequency domain is given by

$$\underline{Y} = \underline{X} \circ \underline{H} + \underline{W}. \tag{1}$$

The symbols  $\underline{X} = [X(0), X(1), ..., X(N-1)]^T$ ,  $\underline{Y} = [Y(0), Y(1), ..., Y(N-1)]^T$ ,  $\underline{H} = [H(0), H(1), ..., H(N-1)]^T$  and  $\underline{W} = [W(0), W(1), ..., W(N-1)]^T$ , are the  $N \times 1$  transmitted signal, received signal, channel frequency response (CFR) and additive white Gaussian noise (AWGN) vectors, respectively. The comb-type pilot pattern is adopted for channel estimation.

After extraction of pilot symbol at the receiver side, the received signal vector at the pilot position can be written as

$$\underline{Y}_p = \underline{X}_p \circ \underline{H}_p + \underline{W}_p \,. \tag{2}$$

The parameters  $\underline{Y}_p$ ,  $\underline{X}_p$  and  $\underline{H}_p$ , are the frequency domain received signal, transmitted signal and CFR vector at the pilot position, respectively. The CFR vector at pilot subcarrier can be represented as

$$\underline{H}_{p} = F_{p}\underline{h} , \qquad (3)$$

where,  $\mathbf{F}_p$  is an  $N \times L$  unitary FFT matrix with  $\mathbf{F}_p(k, l) = e^{-j2\pi k l/N}$ ,  $k = 0 : p_s : (N_p - 1)p_s$  and l = 0 : L - 1. The channel impulse response (CIR) vector is defined as  $\underline{h} = [h(0), h(1), ..., h(L - 1)]$ , with L numbers of independent multipath components. The corresponding autocorrelation of the CIR is given by  $E[\underline{h}\underline{h}^H] = \mathbf{\Lambda}$ , where  $\mathbf{\Lambda} = diag(\Lambda_0, \Lambda_1, ..., \Lambda_{L-1})$ , is the power delay profile (PDP) of the channel and  $\Lambda_l$  denotes the average power of the *l*th delay path.

The channel estimation using LS criterion is given by

$$\underline{\tilde{H}}_{p,ls} = \underline{Y}_p / \underline{X}_p. \tag{4}$$

In order to obtain the channel at all data subcarrier, the interpolation methods are to be performed such as linear interpolation, low pass interpolation, discrete Fourier transform (DFT) based interpolation and so on. In this paper, the DFT based interpolation method is adopted to obtain the CFR at all data subcarrier. After estimating the channel at all data subcarriers, one tap zero forcing equalization is performed to obtain the desired transmitted data signal at the receiver side. The LS channel estimator suffers from high MSE and thus the LMMSE channel estimation is adopted which is optimal in the perspective of mean square error.

From [2], the LMMSE channel estimator at pilot position can be written as

$$\underline{\tilde{H}}_{p,lmmse} = \mathbf{R}_{\underline{H}_p\underline{H}_p} (\mathbf{R}_{\underline{H}_p\underline{H}_p} + (\beta/SNR) \mathbf{I}_{Np})^{-1} \underline{\tilde{H}}_{p,ls}.$$
 (5)

The signal to noise ratio (SNR) is defined as  $SNR = E|\underline{X}_p|^2/\sigma_W^2$ . The symbol  $\beta = E|\underline{X}_p|^2/E|1/|\underline{X}_p|^2$ , is a constant depending on the constellation and  $\sigma_W^2$  is the noise variance.  $\mathbf{I}_{Np}$  is a  $N_p \times N_p$  identity matrix with  $N_p$  as the total number of pilots in a single OFDM system. For quadrature phase shift keying (QPSK) and 16-ary quadrature amplitude modulation (16QAM), the values of  $\beta$  are 1 and 17/9, respectively. The channel frequency autocorrelation matrix is the Fourier transformation of the PDP and is given as

$$\mathbf{R}_{\underline{H}_{p}\underline{H}_{p}} = E[\underline{H}_{p}\underline{H}_{p}^{H}] = \mathbf{F}_{p}\mathbf{\Lambda}\mathbf{F}_{p}^{H}.$$
(6)

The LMMSE estimator experiences high computational complexity of the order  $O(N_p^3)$  due to the matrix inversion operation as given in (6).

#### III. PROPOSED LOW COMPLEXITY LMMSE ESTIMATOR

In this section, an effecticient low complexity LMMSE channel estimator is proposed by exploiting the ciculant matrix properties of the channel frequency autocorrelation matrix. As  $\mathbf{R}_{\underline{H}_p\underline{H}_p}$  and  $\mathbf{R}_{\underline{H}_p\underline{H}_p} + (\beta/SNR)\mathbf{I}_{Np}$ , are circulant matrices and hence their first column defines the whole matrix with the remaining columns are cyclic permutation of the first column [14]. For notational simplicity, let,  $\mathbf{A} = \mathbf{R}_{\underline{H}_p\underline{H}_p}$  and  $\mathbf{B} = \mathbf{R}_{\underline{H}_p\underline{H}_p} + (\beta/SNR)\mathbf{I}_{Np}$ . Thus, the LMMSE estimator is given by

$$\underline{\tilde{H}}_{p,lmmse} = \boldsymbol{A}\boldsymbol{B}^{-1}\underline{\tilde{H}}_{p,ls}.$$
(7)

The A and B matrices are commutative as both the matrices are diagnosable due to circulant matrix property. Thus the LMMSE estimator can be rewritten as

$$\underline{H}_{p,lmmse} = \mathbf{B}^{-1} \mathbf{A} \underline{H}_{p,ls}.$$
(8)

Multiplying both sides of (8) with *B* matrix, the equation (8) becomes

$$\boldsymbol{B}\underline{\boldsymbol{H}}_{p,lmmse} = \boldsymbol{A}\underline{\boldsymbol{H}}_{p,ls}.$$
(9)

By utilizing the ciculant property of both A and B matrices, (9) can be written as

$$\underline{b} \otimes \underline{\tilde{H}}_{p,lmmse} = \underline{a} \otimes \underline{\tilde{H}}_{p,ls}, \tag{10}$$

where,  $\underline{b}$  and  $\underline{a}$  are the first column of **B** and **A** matrices, respectively. From [15], it is know that circular convolution

in frequency domain is same as multiplication in time domain. Thus, after applying inverse fast Fourier transformation (IFFT) algorithm on both sides of (10) and by using circular convolution theorem, the equation becomes

where  $\underline{D} = F_p^H(\underline{a})./F_p^H(\underline{b})$ . The parameters  $F_p^H(\underline{b})$  and  $F_p^H(\underline{a})$ , are the eigenvalues of the corresponding **B** and **A** matrices, respectively. The parameter,  $\underline{\tilde{h}}_{ls}$  and  $\underline{\tilde{h}}_{lmmse}$ , are the estimated channel in the time domain using LS and MMSE criterion, respectively. The estimated channel frequency response is given as  $\underline{\tilde{H}}_{lmmse} = F\underline{\tilde{h}}_{lmmse}$ , where **F** is an unitary  $N \times L$  FFT matrix with the first L columns. The overall computational complexity of the proposed LMMSE estimator is  $O(N_p \log N_p)$ .

### **IV. SIMULATION RESULTS**

In this section, the proposed LMMSE channel estimation method is compared with classical LS and LMMSE channel estimation methods in terms of performance and computational complexity. In this paper, the following parameters are considered for OFDM system model: the number of subcarriers N = 128, length of cyclic prefix  $N_{CP}=16$ , system bandwidth B=1MHz and 16QAM modulation. The exponential decaying model is adopted for developing power delay profile (PDP) of the multipath Rayleigh fading channel, as this is the most popular and widely used channel model [16]. The power of the *l*-th path of the exponential decaying model is given as  $\sigma_l^2 = \sigma_0^2 e^{-l/d}$ . The parameter,  $\sigma_0^2 = 1 - e^{-1/d}/1 - e^{-L/d}$ , is the first multipath component. The symbol  $d = -\tau_{rms}/T_s$ , is the normalized delay spread where  $T_s$  and  $\tau_{rms}$ , are the sampling period and root mean squared (rms) delay of the channel, respectively. The number of multipath channel is given by  $L = \tau_{\max}/T_s$ . The symbol  $\tau_{max}$ , is the maximum excess delay and is defined as  $\tau_{max} = \tau_{rms} \ln A$ , with A being the ratio of non-negligible path power to first path power. In this paper, the value of A, is taken as A = -40 dB and normalized delay spread d = 1.5. This leads to total number of multipath L = 14. The total number of pilots are considered as  $N_p = 16$ , with pilot spacing  $p_s=1:8$ . The total simulation parameters is listed in the Table I. Prefect time and frequency synchronization are assumed at the receiver side.

Fig. 1. shows the MSE performance with respect to SNR for the proposed LMMSE, classical LS and LMMSE methods for normalized delay spread d = 1.5 and multipath channel L = 14. The simulation result shows that the LS estimator has poorer MSE performance as compared to LMMSE estimator due to noise enhancement problem. It is seen that the MSE performance of the proposed LMMSE estimator are exactly matched with the classical LMMSE estimator. This is due to the fact that, the proposed estimator is directly derived from the classical LMMSE estimator without any approximations.

In order to analyze the effect of normalized delay spread d on the performance of channel estimation methods, multiple values of d are taken into considerations e.g. d =

TABLE I Simulation Parameters

Parameters	Value
Number of Subcarriers	128
Number of FFT	128
Number of CP	16
Modulation Type	16QAM
System Bandwidth	1MHz
Subcarrier Spacing	7.8125 KHz
Channel Type	Exponential Decaying PDP
Pilot Spacing	8
Normalized delay spread	1.5
Number of multipath	14



Fig. 1. MSE vs SNR performance comparison of proposed LMMSE, classical LS and LMMSE channel estimator

[0.3 0.6 0.9 1.2 1.5 1.8]. This leads to the number of multipath components as  $L = [3 \ 6 \ 8 \ 12 \ 14 \ 16]$ . The MSE vs normalized delay spread (d) for various channel estimation methods is shown in the Fig. 2. The results are obtained for pilot spacing  $p_s = 8$  at SNR = 25dB. The result shows that the performance of proposed LMMSE is exactly matched with classical LMMSE irrespective of any value of d. It is also observed that the performance of LS is close to LMMSE method for lower value of d. However, the performance gap between the LS and LMMSE method increases with the increase of normalized delay spread value.

The effect of pilot spacing  $p_s$  on the performance of various channel estimation methods are investigated for considering different pilot spacing values e.g.  $p_s = \begin{bmatrix} 2 & 4 & 8 \end{bmatrix}$ .

Fig. 3. shows the MSE vs pilot spacing  $(p_s)$  for various channel estimation methods at SNR = 25dB. The simulation result shows that performance of classical LMMSE and proposed LMMSE are exactly matched irrespective of any  $p_s$  value. It is also observed that the performance of LS is similar to LMMSE for more number of pilot subcarriers. However, The performance gap is increased as fewer numbers of pilots are deployed for channel estimation.

The BER performance comparison of proposed LMMSE, classical LS, LMMSE and perfect channel estimation method is shown in the Fig. 4. The simulation result shows that the



Fig. 2. MSE vs normalized delay spread of proposed LMMSE, classical LS and LMMSE channel estimator



Fig. 3. MSE vs pilot spacing of proposed LMMSE, classical LS and LMMSE channel estimator  $% \mathcal{M} = \mathcal{M} = \mathcal{M} + \mathcal{M} +$ 

performances of proposed and classical LMMSE estimator are exactly same and closer to perfect estimation method where it is assumed that the complete CSI is known at the receiver side. From the results, it is seen that the LS method has similar performance to LMMSE method at very low value of SNR. However, the performance of LMMSE method significantly outperforms the LS method as the SNR value increases.

The impact of the normalized delay spread on BER performance for various channel estimation methods for pilot spacing  $p_s = 4$  and  $p_s = 8$  at SNR = 25dB is shown in the Fig. 5. The simulation result shows that, the performance of LMMSE method is closer to perfect estimation at  $p_s = 4$  and the performance gap increases as the pilot spacing is increased from  $p_s = 4$  to 8. From the simulation result, it is also noticed that, the performance gap between LS and LMMSE methods is small for lower value of d. However, the performance gap between LS and LMMSE methods increases as the value of d increases irrespective of pilot spacing.

Finally, the computational complexity of proposed and



Fig. 4. BER vs SNR performance comparison of proposed LMMSE, classical LS and LMMSE channel estimator



Fig. 5. BER vs normalized delay spread (d) performance comparison of various channel estimation methods at SNR = 25dB

classical LMMSE are compared. In this respect, for a given number of pilot subcarriers  $N_p$ , the proposed LMMSE estimator requires only  $O(N_p \log N_p)$  computational complexity as compared to classical LMMSE estimator which requires  $O(N_p^3)$  number of computations. This is due to the fact that, the proposed LMMSE estimator needs only FFT operation and does not involves in any matrix inversion.

#### V. CONCLUSION

The OFDM is the most promising modulation technique for recent and future wireless communication systems. However, accurate channel estimation is required for efficient use of the OFDM technique. Although, the LMMSE estimation is optimal channel estimation technique in the MSE perspective it requires cubic computational complexity. Therefore, in this paper, a low complexity optimal LMMSE channel estimator has been proposed for OFDM system in frequency selective channel by exploiting the circulant property of the autocorrelation channel frequency response matrix  $R_{HH}$ . Extensive simulation results demonstrate that proposed LMMSE estimator has

exactly the same performance as the classical LMMSE channel estimation method. This is because, the proposed method is derived directly from the classical LMMSE method without any approximations. From the computational complexity perspective, the proposed method does not involve in any matrix inversion rather than FFT operations only. Therefore, it requires only  $O(N_p \log N_p)$  complexity as compared to classical LMMSE method which requires  $O(N_p^3)$  complexity where  $N_p$  is the total number of pilot subcarriers.

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