Performance Study of a Class of Irregular Near Capacity Achieving LDPC Codes

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Abstract—**This paper investigates the performance of a class of irregular low-density parity-check (LDPC) codes through a recently published low complexity upper bound on their belief-propagation decoding thresholds. Moreover, their performance analysis is carried out through a recently published algorithmic method, presented in Babich et al. 2017 paper. In particular, the class considered is characterized by variable node degree distributions $\lambda(x)$ of minimum degree $i_1 > 2$: being, in this case, $\lambda'(0) = \lambda_2 = 0$, this is useful to design LDPC codes presenting a linear minimum distance growth with the block length with probability 1, as shown in Di et al.'s 2006 paper. These codes unfortunately cannot reach capacity under iterative decoding, since the achievement of capacity requires $\lambda_2 \neq 0$. However, in this latter case, the block error probability might converge to a constant, as shown in the aforementioned paper.**

I. Introduction

LDPC codes are a class of channel block codes, first introduced in the 1960's by Robert Gallager [1], representing the leading edge in modern channel coding. Due to the technical limitations of that age, LDPC codes were scarcely considered for almost 30 years, apart from Tanner's generalized LDPC definition and graphical representation, presented in his 1981 paper [2] (which was later called Tanner graph), and were re-invented in the mid 1990's by MacKay [3] and Luby et al. [4]. After being included in modern communication standards such as digital video broadcasting DVB-S2 (satellite communication), ITU-T G.hn (home networking), and DOCSIS 3.1 (cable) standards, they are also used in the IEEE802.11 (Wi-Fi) [5], [6], 802.16e (Wi-MAX), and 10G-BaseT Ethernet standards, and have been also proposed as component codes of product code structures [7] for the next generation digital terrestrial broadcasting transmission system [8]. Moreover, they were recently adopted, together with polar codes, by the fifth-generation (5G) new radio (NR) standard (see, e.g., [9] - [11]).

As first noticed by Gallager in his aforementioned introductory work, limited to regular LDPC codes, these exhibit the so called "threshold phenomenon". Namely, an upper bound for the channel noise can be defined by the noise threshold so that, if the channel noise is maintained below this threshold, the probability of lost information can be made as small as desired. Later, Luby et al. showed that *irregular* LDPC codes perform better than regular ones [4], and exhibit the threshold phenomenon, too. In this work, they also showed that their hard-decision decoding process may be analyzed considering an individual edge between a variable node m and a check node c and an associated tree, rooted in m, describing its neighborhood, as shown in Fig. 1 of [4]. Denote with p_l the probability that m sends c an incorrect value at the l-th iteration. Following the work of Gallager [1], in [4] a recursive equation describing the evolution of p_l was determined over a constant number of iterations.

LDPC codes are capacity-approaching codes, which means that practical constructions exist that allow the noise threshold to be set very close to the theoretical maximum (the Shannon limit) for a symmetric memoryless channel. Thus, the problem of an easy evaluation of the threshold, and, in general, of the performance of belief propagation decoding is important to allow the design of capacity-approaching codes, based on noise threshold maximization (see, e.g., [12] where, using the analysis outlined in [13], a custom software based on [14] was employed to simulate the performances of punctured LDPC codes over an additive white Gaussian noise (AWGN) channel, assuming a binary phase shift keying modulator).

Our paper [15] was focused on the investigation about the usefulness of a low complexity upper bound on belief-propagation decoding thresholds, recently published in [16], for the class of irregular LDPC codes characterized by variable node degree distributions

$$\lambda(x) = \sum_{i=i_1}^{d_l} \lambda_i x^{i-1}$$

of minimum degree $i_1 > 2$, being λ_i the fraction of edges in the Tanner graph connecting to a degree-i variable node and d_l the maximum variable node degree. This investigation was absolutely novel. In fact, the above mentioned upper bound of [16][1] was conceived as an algebraic method to calculate

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[1]In [16], three low complexity upper bounds to the exact belief-propagation decoding thresholds were derived, by applying the result of [20] to the asymptotical behavior of $\Delta(s, t)$, defining the difference between the mean values of a generic check node output messages at the l-th and at the $(l-1)$-th iteration. As shown in [16], the second and third bounds, therein presented, called s^*_{approx} and s^*_{Jensen}, respectively, are applicable only to a class of LDPC codes, namely, to the class of codes presenting variable node degree distributions $\lambda(x)$ of minimum degree $i_1 = 2$. The first bound found in [16], instead, called s^*_{bound}, is more generically valid $\forall i_1 \geq 2$.

upper bounds on belief-propagation decoding thresholds for the class of irregular LDPC codes characterized by $i_1 \geq 2$, since, in the literature (see, e.g., [17] and [18]), only the case with $i_1 = 2$ was considered before, as far as the upper bounds on thresholds and on the coefficient λ_{i_1} (stability condition) are concerned (see Example 12 of [18][2]).

In the present paper, in addition to what already presented in [15], we investigate the suitability of a recently published algorithmic method, presented in [20][3], to find the convergence conditions of the above mentioned recursive sequence describing the evolution of p_l in [4]. This investigation is motivated by the need of analyzing the class of irregular near capacity achieving LDPC codes therein considered, characterized by variable node degree distributions $\lambda(x)$ of minimum degree $i_1 > 2$. This analysis is completely novel since, so far, the performance analysis we conducted in [21] and [22] was focused on the class of LDPC codes presenting degree distributions $\lambda(x)$ of minimum degree $i_1 = 2$. In order to accomplish this task, we investigated on the role played by the product $\lambda'(0)\rho'(1)$ in determining this performance, being $\rho(x) = \sum_{j=2}^{d_r} \rho_j x^{j-1}$, ρ_j the fraction of edges connecting to a degree-j check node, and d_r the maximum check node degree. However, as shown in [21], being $\lambda'(0) = \lambda_2$, this analysis is feasible only for the class of LDPC codes presenting $\lambda_2 \neq 0$.

The investigation on the class of irregular LDPC codes characterized by variable node degree distributions $\lambda(x)$ of minimum degree $i_1 > 2$ is interesting because the fact that, in this case, $\lambda'(0) = \lambda_2 = 0$, implies that these codes present a linear minimum distance growth with the block length with probability 1, as shown in [23]. These codes unfortunately cannot reach capacity under iterative decoding, since the achievement of capacity requires $\lambda_2 \neq 0$. However, in this latter case, the block error probability might converge to a constant [23], since, as shown in [24], in order to have zero word error probability, it is necessary to have $\lambda_2 = 0$. This was proved in [24] by the following argument: if $\lambda_2 > 0$, then in the ensemble, as the block length $n \to \infty$, the average number of weight 2 codewords is bounded away from zero. Hence even a maximum likelihood decoder would have non-zero decoding error probability, fact that does not happen with $\lambda_2 = 0$.

The paper is organized as follows. In Section II we recall the first upper bound to the belief propagation decoding thresholds already derived in [16], since the paper is focused on it. In Section III, the method outlined in [4] for designing irregular graphs is recalled, based on a recursive equation describing the evolution of the above mentioned p_l, for a given value of the parameter p_0. This is necessary so as to provide the reader with a reference to the mathematical functions needed in the following. In Section IV we rewrite the mathematical method presented in [20] in a form suitable to obtain the "supremum p^* of all values of p_0 for which the sequence p_l is monotonically decreasing and hence converges to 0" [4]. In Section V some numerical results are given to investigate the usefulness of the above mentioned bound in determining the performance of an irregular LDPC code characterized by variable node degree distributions $\lambda(x)$ of minimum degree $i_1 > 2$. Moreover, some numerical results are also given in terms of the obtainable p^* values for the class of irregular LDPC codes considered in [4] reporting some simulation results, too. Finally, Section VI summarizes the results of the paper.

[2]As far as the LDPC codes found in [18] are concerned, all presenting variable node degree distributions $\lambda(x)$ of minimum degree $i_1 = 2$, in [16] and [19] were presented the upper bounds to their exact belief-propagation decoding thresholds.

[3]The method consists in solving a problem of quadratic degeneracy instead of looking for the conditions guaranteeing the convergence of a certain sequence, task which normally has to be performed manually, i.e., by repeated trials.

II. Upper Bounds on LDPC Codes Decoding Thresholds

To determine the upper bounds on thresholds, first of all we have to determine the asymptotical behaviour of (18) in [15] for $t \to \infty$. The first upper bound, called s^*_{bound} in [16], was found computing $\phi(x)$, defined in (9) of [15], with $x \geq 10$. To this end, we have added a further invertible approximation of the function $\phi(x)$, the derivation of which is given in the Appendix of [16]. To obtain the second upper bound, called s^*_{approx} in [16], we have used the approximation (16) of [16] (that was implicitly used in [17]). The third upper bound, called s^*_{Jensen} in [16], was obtained applying the Jensen's inequality to the latter asymptotical approximation.

A. Upper bound on LDPC codes decoding thresholds holding for $i_1 \geq 2$

As far as the above mentioned first upper bound on LDPC codes thresholds (called s^*_{bound} in [16]) is concerned, given a degree distribution $\lambda(x)$ of minimum degree $i_1 \geq 2$, defining $z(s,t) := \frac{s+(i_1-1)t}{2}$ and $A_j := \frac{1}{(j-1)^2\lambda_{i_1}^2}$, the following lemma is proved in [26].

Lemma: As $t \to \infty$, being $W(\cdot)$ the Lambert-W function:

$$f(s,t) = 2\sum_{j=2}^{d_r} \rho_j W(A_j z(s,t)\,\mathrm{e}^{z(s,t)}) + O(t^{-1}) \tag{1}$$

With this $f(s,t)$ and remembering that $\frac{\mathrm{d}W(x)}{\mathrm{d}x} = \frac{1}{x+e^{\mathrm{W}(x)}}$:

$$f_t(s,t) = 2\sum_{j=2}^{d_r} \rho_j \frac{z_t(s,t)e^{z(s,t)}(1+z(s,t))}{z(s,t)e^{z(s,t)} + e^{\mathrm{W}(A_j z(s,t)e^{z(s,t)})-\log A_j}} \tag{2}$$

Applying (15) of [15] to (1) and (2) we get:

$$\begin{cases} 2\sum_{j=2}^{d_r} \rho_j \mathrm{W}(A_j z(s,t)e^{z(s,t)}) = t \\ 2\sum_{j=2}^{d_r} \rho_j \frac{z_t(s,t)e^{z(s,t)}(1+z(s,t))}{z(s,t)e^{z(s,t)}+e^{\mathrm{W}(A_j z(s,t)e^{z(s,t)})-\log A_j}} = 1 \end{cases} \tag{3}$$

and (18) of [15] can be rewritten as:

$$\begin{cases} 2\sum_{j=2}^{d_r} \rho_j \mathrm{W}(A_j z(s,t)e^{z(s,t)}) - t = 0 \\ 2\sum_{j=2}^{d_r} \rho_j \frac{z_t(s,t)e^{z(s,t)}(1+z(s,t))}{z(s,t)e^{z(s,t)}+e^{\mathrm{W}(A_j z(s,t)e^{z(s,t)})-\log A_j}} - 1 = 0 \end{cases} \tag{4}$$

Its solution (s^*_{bound}, t^*_{bound}), obtained applying the instruction set produced in [20], determines the bound $\sigma^*_{\text{bound}} = \sqrt{\frac{2}{s^*_{\text{bound}}}}$, which is valid $\forall i_1$, unlike the other two (s^*_{approx} and s^*_{Jensen}) reported in [16], which hold both for $i_1 = 2$ only.

III. Irregular Graphs Design

Irregular LDPC codes [18] are defined by specifying the distribution of the node degrees in their Tanner graphs. In particular, in the edge-perspective degree distribution, λ_i is the fraction of edges in the Tanner graph connecting to a degree-i variable node, and ρ_j is the fraction of edges connecting to a degree-j check node.

Consider an irregular LDPC code with edge-perspective degree distributions $\lambda(x)$ and $\rho(x)$, defined as

$$\lambda(x) := \sum_{i=i_1}^{d_l} \lambda_i x^{i-1} \tag{5}$$

$$\rho(x) := \sum_{j=2}^{d_r} \rho_j x^{j-1} \tag{6}$$

being d_l (respectively d_r) the maximum variable (respectively check) node degree. The d_l-tuple $\{\lambda_i\}$ and d_r-tuple $\{\rho_j\}$ both add up to 1.

Gallager's hard-decision decoding approach [1] has been generalized in [4] to the case of irregular graphs, in order to consider the varying degrees of the variable and check nodes. It may be analyzed considering an individual edge between a variable node m and a check node c and an associated tree, rooted in m, describing its neighborhood, as shown in Fig. 1 of [4]. Denote with p_l the probability that m sends c an incorrect value at the l-th iteration. Following the work of Gallager, in [4] is determined a recursive equation describing the evolution of p_l over a constant number of iterations of the message passing decoding algorithm used for hard-decision decoding. Given an irregular LDPC code with given distributions (5) and (6), and fixing a p_0 value, this recursive equation is simply given by

$$p_l = f(p_0, p_{l-1}) \tag{7}$$

where, for $0 \leq p_0 \leq 1$ and $0 \leq p \leq 1$, the function $f(p_0, p)$ is defined as

$$f(p_0, p) := p_0 - \sum_{i=i_1}^{d_l} \lambda_i f_i(p_0, p) \tag{8}$$

through $f_i(p_0, p)$, defined as

$$\begin{aligned} f_i(p_0,p) := \quad & p_0 \sum_{k=b_i(p_0,p)}^{i-1} \binom{i-1}{k} \left[\frac{1+\rho(1-2p)}{2}\right]^k \\ & \cdot \left[\frac{1-\rho(1-2p)}{2}\right]^{i-1-k} \\ & +(1-p_0) \sum_{k=b_i(p)}^{i-1} \binom{i-1}{k} \left[\frac{1-\rho(1-2p)}{2}\right]^k \\ & \cdot \left[\frac{1+\rho(1-2p)}{2}\right]^{i-1-k} \end{aligned} \tag{9}$$

being $b_i(p_0, p)$ defined as

$$b_i(p_0,p) := \left\lceil \frac{1}{2}\left(i - 1 + \frac{\log((1-p_0)/p_0)}{\log((1+\rho(1-2p))/(1-\rho(1-2p)))}\right) \right\rceil \tag{10}$$

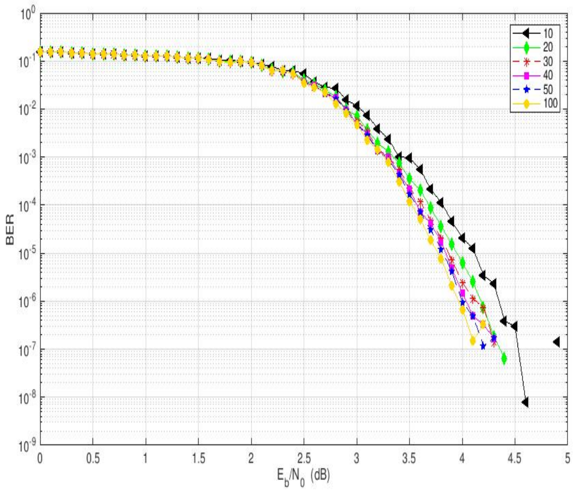

Fig. 1. BER vs. E_b/N_0 in dB of the rate-1/2 randomly chosen codes with distribution pairs (λ, ρ) given in (13), called *Code 22* in [4]. The performance curves have been obtained with an iteration number I = 10, 20, 30, 40, and 50 of the decoding algorithm.

Gallager's idea, resumed in [4], is then to find the supremum p^* of all values of p_0 for which the sequence p_l is monotonically decreasing and hence converges to 0. He also proves that, as the block length of the code and girth of the graph grow large, this decoding algorithm works for all $p_0 < p^*$.

IV. Low Complexity Determination of p^*

Applying the method defined in [20], instead of searching the last value of the parameter p_0 granting the convergence of (7), we solve a problem of quadratic degeneracy which can be assigned to a standard software.

When the second partial derivative of $f(p_0, p)$ with respect to p, $f_{pp}(p_0, p)$, is $\neq 0$, the problem of quadratic degeneracy is the system of equations

$$\begin{cases} f(p_0, p) = p \\ f_p(p_0, p) = 1 \end{cases} \tag{11}$$

where $f_p(p_0, p)$ is the first partial derivative of $f(p_0, p)$ with respect to p.

The solution of (11) gives the value p^*, namely the maximum p_0 granting the convergence of (7).

V. Numeric Results

With regard to irregular LDPC codes with $i_1 > 2$, as example we consider the rate-1/2 irregular LDPC codes given in [4] with $i_1 = 5$, having degree distributions:

$$\begin{aligned} \lambda(x) = \quad & 0.496041x^4 + 0.173862x^5 + 0.077225x^{20} \\ & +0.252871x^{22} \\ \rho(x) = \quad & x^{13} \end{aligned} \tag{12}$$

and

$$\begin{aligned} \lambda(x) = \quad & 0.284961x^4 + 0.124061x^5 + 0.068844x^{26} \\ & +0.109202x^{28} + 0.119796x^{29} + 0.293135x^{99} \\ \rho(x) = \quad & x^{21} \end{aligned} \tag{13}$$

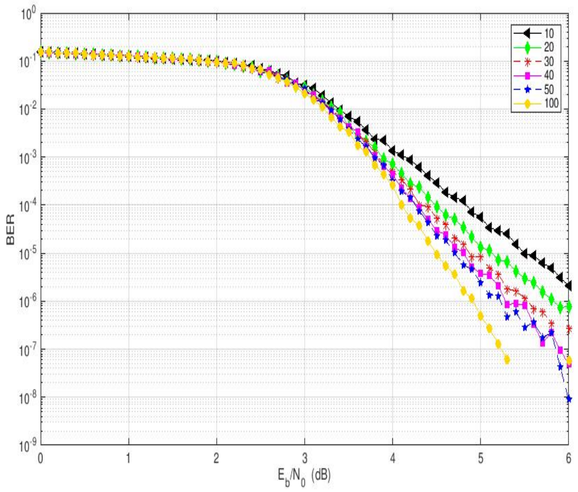

Fig. 2. BER vs. E_b/N_0 in dB of the rate-1/2 randomly chosen codes with distribution pairs (λ, ρ) given in (12), called *Code 14* in [4]. The performance curves have been obtained with an iteration number I = 10, 20, 30, 40, and 50 of the decoding algorithm.

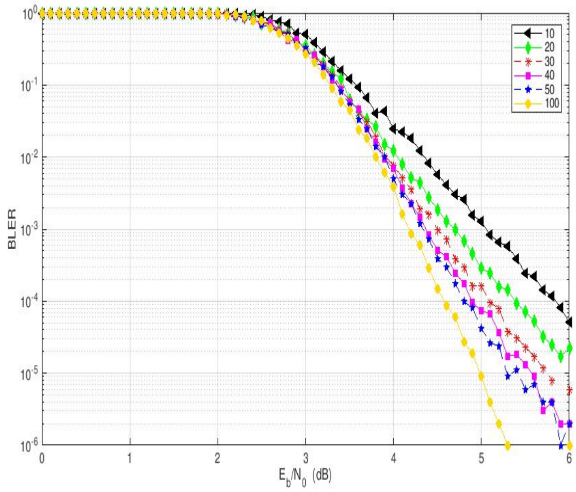

Fig. 4. BLER vs. E_b/N_0 in dB of the rate-1/2 randomly chosen codes with distribution pairs (λ, ρ) given in (12), called *Code 14* in [4]. The performance curves have been obtained with an iteration number I = 10, 20, 30, 40, and 50 of the decoding algorithm.

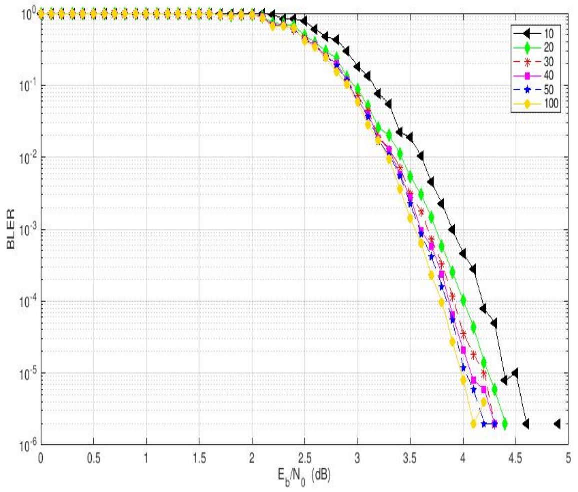

Fig. 3. BLER vs. E_b/N_0 in dB of the rate-1/2 randomly chosen codes with distribution pairs (λ, ρ) given in (13), called *Code 22* in [4]. The performance curves have been obtained with an iteration number I = 10, 20, 30, 40, and 50 of the decoding algorithm.

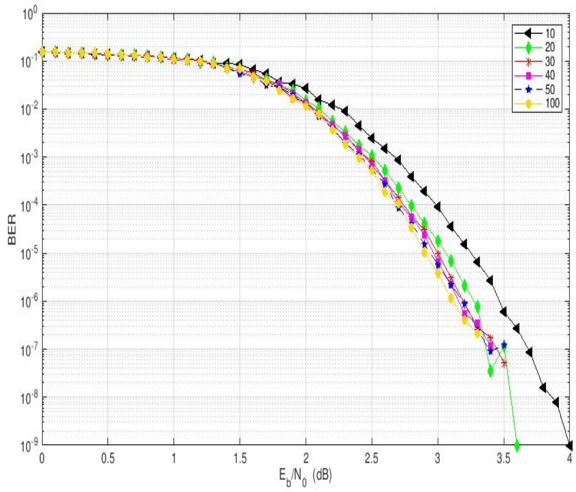

Fig. 5. BER vs. E_b/N_0 in dB of the rate-1/2 randomly chosen codes with distribution pairs (λ, ρ) given in (15), called *Code 14'* in [4]. The performance curves have been obtained with an iteration number I = 10, 20, 30, 40, and 50 of the decoding algorithm.

and with $i_1 = 3$, having degree distributions:

$$\begin{aligned} \lambda(x) = &\quad 0.123397x^2 + 0.555093x^3 + 0.321510x^{15} \\ \rho(x) = &\quad x^9 \end{aligned} \tag{14}$$

and

$$\begin{aligned} \lambda(x) = &\quad 0.093368x^2 + 0.346966x^3 + 0.159355x^{20} \\ &\quad +0.400312x^{22} \\ \rho(x) = &\quad x^{13} \end{aligned} \tag{15}$$

The first two codes (12) and (13), called *Code 14* and *Code 22* in [4], respectively, both present variable node degree distributions $\lambda(x)$ of minimum degree $i_1 = 5$ and, therefore, this ensures that their graphs have good expansion, as proved in Lemma 3 of [4]. In fact, in this Lemma 3 it is shown that, when $i_1 \geq 5$, the polynomial $\rho(x)$ has degree at most J for some constant J. Secondly, it is demonstrated that, given the block length n, with probability $1 - O(1/n)$, for some fixed $\alpha > 0$, $\epsilon > 0$, and $\beta = 3/4 + \epsilon$, the bipartite graph of the code is an (α, β) expander. Thus, the analysis performed in [26] to determine the asymptotical behaviour of (18) in [15] for $t \to \infty$ can be applied (to find the upper bounds $\sigma^*_{\text{bound}} = \sqrt{\frac{2}{s^*_{\text{bound}}}}$, solution of (4)). Moreover, since, for the codes considered, $i_1 > 2$, the bounds s^*_{approx} and s^*_{Jensen}, reported in [16], cannot be applied.

On the other hand, as far as the other two codes (14) and

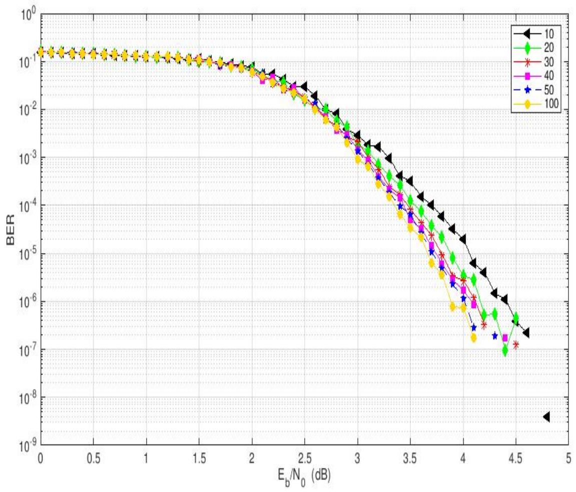

Fig. 6. BER vs. E_b/N_0 in dB of the rate-1/2 randomly chosen codes with distribution pairs (λ, ρ) given in (14), called *Code 10'* in [4]. The performance curves have been obtained with an iteration number I = 10, 20, 30, 40, and 50 of the decoding algorithm.

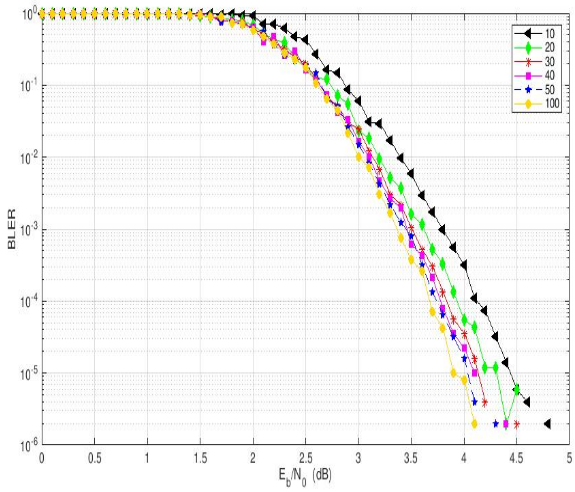

Fig. 8. BLER vs. E_b/N_0 in dB of the rate-1/2 randomly chosen codes with distribution pairs (λ, ρ) given in (14), called *Code 10'* in [4]. The performance curves have been obtained with an iteration number I = 10, 20, 30, 40, and 50 of the decoding algorithm.

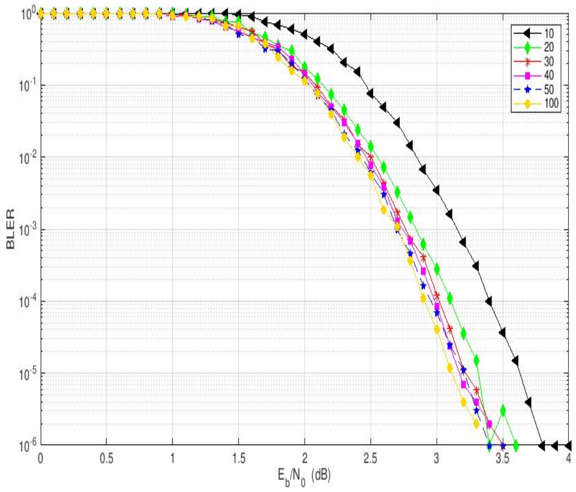

Fig. 7. BLER vs. E_b/N_0 in dB of the rate-1/2 randomly chosen codes with distribution pairs (λ, ρ) given in (15), called *Code 14'* in [4]. The performance curves have been obtained with an iteration number I = 10, 20, 30, 40, and 50 of the decoding algorithm.

(15) are concerned, called *Code 10'* and *Code 14'* in [4], both present variable node degree distributions $\lambda(x)$ of minimum degree $i_1 = 3$ and have associated bipartite graphs that do not have a sufficient expansion (for Lemma 3 in [4] to hold). Thus, our above mentioned Lemma (1) cannot instead be applied.

The two codes with degree distribution polynomials (12) and (13), respectively, present a threshold upper bound $\sigma^*_{\text{bound}} = 1.02239$ and $\sigma^*_{\text{bound}} = 1.07602$, respectively. The Mathematica® script written to determine these results is reported in Appendix A of [15].

In [4], after defining p_l as the probability that a variable node sends an incorrect message in round l of a message passing decoding algorithm, the authors have determined, for each of the four rate-1/2 codes reported in Table I of [4], also the values p^*, defined as the "supremum of all values of p_0 for which the sequence p_l is monotonically decreasing and hence converges to 0". The two codes with degree distribution polynomials (12) and (13), respectively, have p^* values of 0.0505 and 0.0533, respectively, when the transmission of coded symbols over a binary-symmetric channel is considered, whereas the two codes with degree distribution polynomials (14) and (15), respectively, have p^* values of 0.0578 and 0.0627, respectively.

Solving (11), instead, we found that the two codes with degree distribution polynomials (12) and (13), respectively, have p^* values of 0.05053 and 0.05334, respectively, whereas the two codes with degree distribution polynomials (14) and (15), respectively, have p^* values of 0.05781 and 0.06272, respectively, thus obtaining a perfect agreement with the results of [4].

As noticed in [4], "p^* represents the error rate we would expect to be able to handle for arbitrarily long block lengths, and that we only expect to approach p^* asymptotically in practice as the number of nodes grows". Thus, higher p^* values lead to a better code performance, as shown in Figs. 3 and 4 of [4], where the experimental results with hard decision decoding are presented. As shown in Fig. 3 of [4], *Code 22*, presenting a p^* value of 0.0533, performs better than *Code 14*, presenting a p^* value of 0.0505, and, similarly, as shown in Fig. 4 of the same paper, *Code 14'*, presenting a p^* value of 0.0627, performs better than *Code 10'*, presenting a p^* value of 0.0578.

As expected, and as noted also in [18], where σ^* and p^* values have been compared in Tables I and II, the code characterized by a lower p^* value, i.e., the code with degree distribution polynomials (12), also presents a lower threshold upper bound σ^*_{bound} value.

A. Simulation Results

Consider an *ensemble* of random codes with edge-perspective degree distributions $\lambda(x)$ and $\rho(x)$ given above. A custom software based on [14] (also used in [13] and [25] to design well performing rate compatible puncturing patterns for LDPC codes on the basis of the results of [16] and [20]) was employed to simulate their performance over an additive Gaussian noise channel (AWGN) channel, assuming a binary phase shift keying (BPSK) modulator. The belief propagation algorithm, also called message passing or sum-product algorithm, commonly employed for LDPC decoding, has been adopted, employing soft decision.

In Figs. 1 and 2 are shown the bit error rate (BER) performance curves of some randomly chosen codes with distribution pairs (λ, ρ) given in (12) and (13), called *Code 14* and *Code 22* in [4], respectively. As expected from the hard decoding experiments conducted in [4] and mentioned above, the BER performance obtained using *Code 22* is better than that obtained with *Code 14*, since the first presents a higher p^* value. As far as the block error rate performance (BLER) is concerned, reported in Figs. 3 and 4, the same conclusions still hold.

In Figs. 5 and 6 are shown the bit error rate (BER) performance curves of some randomly chosen codes with distribution pairs (λ, ρ) given in (15) and (14), called *Code 14'* and *Code 10'* in [4], respectively. As expected from the hard decoding experiments conducted in [4] and mentioned above, the BER performance obtained using *Code 14'* is better than that obtained with *Code 10'*, since the first presents a higher p^* value. As far as the block error rate performance (BLER) is concerned, reported in Figs. 7 and 8, the same conclusions still hold.

VI. Conclusions

Owing to their good performance, LDPC codes are an important family of error-correction codes employed in current data communication systems. In this paper, the analysis of irregular LDPC codes based on some low complexity upper bounds on their belief-propagation decoding thresholds, recently presented in [16], was addressed. This was possible thanks to the work in [17], upon which the derivation is based, and to the algorithmic method for LDPC codes threshold evaluation proposed in [20]. The results found in the paper show that the first low complexity upper bound on belief-propagation decoding thresholds, published in [16], is useful for the class of irregular low-density parity-check (LDPC) codes for which it was conceived, i.e., for the LDPC codes characterized by variable node degree distributions $\lambda(x)$ of minimum degree $i_1 > 2$. However its validity is restricted to bipartite graphs that present a sufficient expansion, namely to the bipartite graphs of LDPC codes fulfilling the conditions given in Lemma 3 of [4]. The analysis published in [16] cannot be applied, instead, to bipartite graphs that do not have a sufficient expansion. Furthermore, we investigated the suitability of the algorithmic method presented in [20] also to carry out the performance analysis of the class of irregular near capacity achieving LDPC codes considered. The p^* values, representing the error rate we would expect to be able to handle for arbitrarily long block lengths, obtained applying the quadratic degeneracy theory, have been shown to be in perfect agreement with those reported in [4]. Moreover, the simulation results obtained applying the belief propagation algorithm, employing soft decision, have been shown to be in perfect agreement with the hard decoding experimental results reported in the same paper.

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