Parameter Estimation of Weibull Distribution Based on Second-Kind Statistics

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Abstract: **The log-cumulant estimator is proposed to estimate the parameters of Weibull distribution based on second-kind statistics. With the explicit closed form expressions, the log-cumulant estimator is computationally efficient. Parameter estimation results from Monte Carlo simulation and real synthetic aperture radar (SAR) image demonstrate that the log-cumulant estimator leads to better performance when compared to the moment estimator.**

Index Terms: **Weibull distribution, parameter estimation, second-kind statistics, log-cumulant estimator**

I. Introduction

The Weibull distribution has been widely applied to synthetic aperture radar (SAR) images of sea, land, weather, and sea-ice clutter, and it contains the classical Rayleigh and exponential distributions as special cases [1-5]. With two parameters (shape parameter and scale parameter), the Weibull distribution can fit the experimental data better than the one-parameter distributions such as Rayleigh [1, 6]. For example, the Rayleigh distribution describes the early low-resolution SAR images well enough, but for the higher resolution SAR images, the two-parameter Weibull distribution can characterize the image contrast precisely [1].

In order to use the Weibull model in practical applications, its parameters should be estimated accurately. The estimation methods of Weibull distribution are summarized in [7], including linear estimator, maximum likelihood estimator, moment estimator, and Bayesian estimator. The linear estimator is the linear combinations of order statistics with suitably chosen coefficients. However, the determination of the coefficients is very difficult owing to a huge amount of computation, so it usually requires table look-ups. The maximum likelihood estimator is the parameter value that maximizes the likelihood function, given the data available. However, the maximum likelihood estimator has to solve the nonlinear equation, which usually requires the iterative method. The moment estimator estimates the parameters by directly using the statistical moments of Weibull distribution, but it does not have the explicit closed form and needs some numerical optimization techniques. The Bayesian estimator is the value of parameter that maximizes the posterior density in terms of the Bayesian theorem. However, the Bayesian estimator requires the prior distribution, which is not easy to determine.

In this letter, the log-cumulant estimator is proposed for the Weibull distribution based on second-kind statistics, which relies on the Mellin transform [8, 9]. We compare the log-cumulant estimator with the moment estimator, and we have observed that the performance of the moment estimator is degraded seriously for the small values of the shape parameter, but the log-cumulant estimator leads to high estimation accuracy no matter what values are chosen for the shape parameter, which is validated by parameter estimation results from Monte Carlo simulation and real SAR image experiment. Consequently, we recommend the log-cumulant estimator instead of the moment estimator.

This letter is organized as follows. The Weibull distribution is introduced in Section II. The moment estimator is briefly introduced in Section III, and the log-cumulant estimator based on second-kind statistics is proposed in Section IV, including the derivation process, Monte Carlo simulations, and real SAR image experiment. Lastly, this letter is concluded in Section V.

II. Weibull Distribution

The Weibull distribution has the following probability density function (pdf) [1]

$$f_{c,b}(x)=\frac{cx^{c-1}}{b^c}\exp\left[-\left(\frac{x}{b}\right)^c\right],\quad x\geq 0\,, \tag{1}$$

where c ($c>0$) is the shape parameter and b ($b>0$) is the scale parameter. The appearance of Weibull pdf is determined by the shape parameter c. When $c<1$, the pdf curve is J-shaped. When $c>1$, the pdf curve becomes skewed unimodal [7]. Denoting X as the Weibull-distributed random variable with parameters c and b, it can be demonstrated that the new random variable $\frac{X}{b}$ is still Weibull-distributed with the shape parameter c and the unit scale parameter ($b=1$).

Manuscript received June 12, 2010, revised September 4, 2010.

This work has been supported by the State Key Development Program for Basic Research of China (Grant No. 2007CB311006), the National High Technology Research and Development Program of China (Grant No. 2006AA01Z126), the National Natural Science Foundation of China (Grant No. 60602026), and the Specialized Research Fund for the Doctoral Program of Higher Education of China (Grant No. 20070698002).

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This means that the Weibull distribution can be readily normalized. For various values of c, the pdf of Weibull distribution is plotted in Fig. 1. Obviously, the value of c controls the shape of the pdf. It should be noted that the Weibull distribution reduces to the exponential distribution when $c=1$ and to the Rayleigh distribution when $c=2$.

The Weibull distribution can be simulated by [7]

$$X = b\left[-\log\left(Y\right)\right]^{1/c}, \tag{2}$$

where X is the Weibull random variable with shape parameter c and scale parameter b, and Y is the random variable uniformly distributed in the interval $\left(0,1\right)$. With the help of (2), the Weibull-distributed samples can be simulated, which are shown in Fig. 2 for various values of c. It is apparent that the Weibull samples with $c=0.15$ show much severer impulsiveness than the ones with $c=2$. In general, the smaller the value of c is, the more impulsive the Weibull samples are. Since the Weibull-distributed samples can be simulated readily, we can use the Monte Carlo simulation to compare the performance of various parameter estimators.

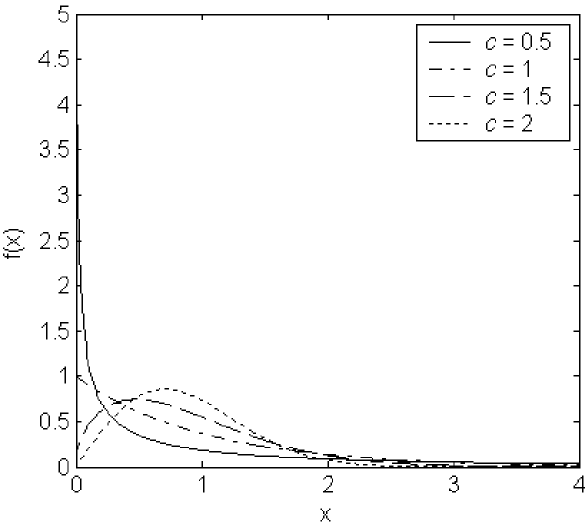
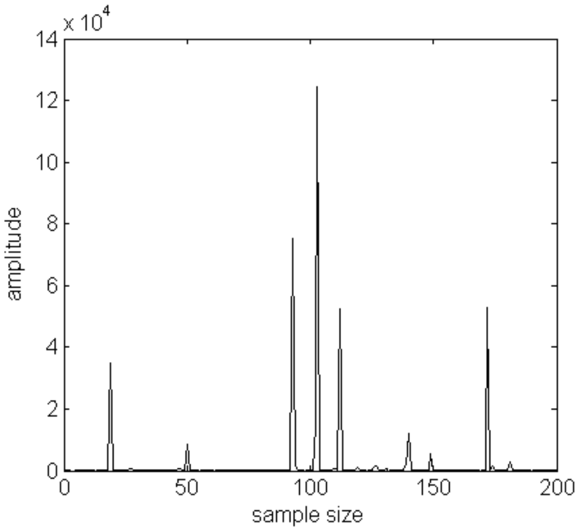
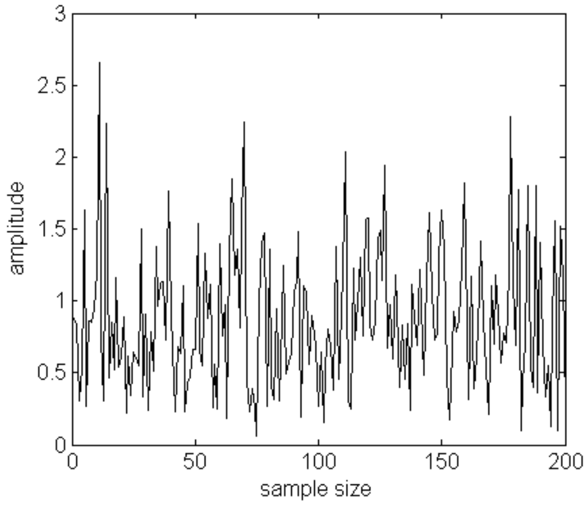


Fig. 1. Pdfs of Weibull distribution ($b=1$)



(a) $c=0.15$



(b) $c=2$

Fig. 2. Weibull-distributed samples ($b=1$ and the number of samples is 200)

III. Moment Estimator

The n th order moment of Weibull distribution can be written as

$$E\left(X^n\right) = b^n\Gamma\left(1+\frac{n}{c}\right), \quad n=1,2,\ldots, \tag{3}$$

where X is the Weibull random variable with parameters c and b, and $\Gamma\left(\cdot\right)$ is the Gamma function. Hence, the moment estimator for the Weibull distribution is straightforward as follows [1, 7]:

$$\frac{E\left(X^2\right)}{E^2\left(X\right)} = \frac{\Gamma\left(1+2/c\right)}{\Gamma^2\left(1+1/c\right)}, \tag{4}$$

$$E\left(X\right) = b\Gamma\left(1+\frac{1}{c}\right). \tag{5}$$

By replacing the actual moments with the sample moments, parameters c and b can be subsequently estimated from (4) and (5), using some numerical optimization techniques such as bisection [10].

The moment estimator was tested for various true values of parameter c according to Monte Carlo simulation. The Weibull-distributed samples were simulated independently by using (2), and the number of samples is 10000. For each true parameter c, the Monte Carlo simulation experiment was repeated 100 times independently, and then the average and standard deviation of the estimates were computed. The results are shown in Table I with standard deviations in parentheses. Obviously, the performance of the moment estimator relies on the true values of c. For the larger values of c, the moment estimator can lead to high estimation accuracy (e.g., $c=2$). However, if the smaller values are chosen for the c (e.g., $c=0.15$), the moment estimator results in poor performance. In other words, the moment estimator is sensitive to samples. If the samples show severe impulsiveness, which corresponds to the small values of the shape parameter (e.g., $c=0.15$), the moment estimator cannot achieve high estimation accuracy.